43rd IEEE Conference on Decision and Control
December 14-17, 2004
Atlantis, Paradise Island, Bahamas
ThA08.3

On stabilization of linear systems with stochastic disturbances and input saturation

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***Abstract*— It is well-known that for linear systems internal asymptotic stability implies external stability in the sense that when the external input is in L_p then also the state will be in L_p. However, for the control of linear systems with saturation where the controlled system is nonlinear this implication is no longer directly applicable. Several people have studied the effect of external inputs in L_p either directly or in the context of ISS as introduced by Sontag. In this paper we will study the effect of external stochastic disturbances on linear systems with input saturation and we establish that when we can achieve internal global asymptotic stability then we can also achieve a bounded variance for the state.**

I. INTRODUCTION

In this paper we consider the problem of stabilizing a linear system which is subject to input saturation and stochastic disturbances. The question of global stabilization of a linear system subject to input saturation has deserved a great deal of attention in the literature. Within the continuous time setting of this problem, it is known that global asymptotic stabilization can be achieved through bounded inputs if and only if all poles of the open loop system have non-positive real parts. In that case, in general, stabilization will not be possible through linear feedbacks. See, for example [7], [2], [11].

For discrete time systems similar conclusions hold. Indeed, global stabilization of a linear discrete time system subject to amplitude constrained inputs can be achieved if and only if the poles of the open loop system lie inside or on the unit circle of the complex plane. Moreover, the class of linear feedbacks is, in general, too restrictive for achieving global asymptotic stability. See, for example [12], [4].

Variations of this theme include an analysis of the question of what can be achieved by saturated linear feedbacks. In [3] this question has been partially answered by introducing the notion of semi-global stabilization and using low-gain linear feedbacks.

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The notion of global stabilization in these papers requires the state to vanish as time converges to infinity, irrespective of the initial condition of the system. As such, this notion of stability is an *internal* one as it ignores the effect that external disturbances may have on the state. An interesting generalization therefore involves the question when *external stability* can be achieved through bounded inputs. By this, we mean that for zero initial condition the state of the controlled system belongs to the Lebesgue space L_p (or ℓ_p) whenever the external disturbance belongs to L_p (ℓ_p). Of course, for unconstrained systems internal stability implies external stability. However, for systems subject to input constraints the answers are more delicate. We refer to [6], [10]. Note that in this context also the ISS concept of Sontag (see for instance [8], [9]) has been studied in quite some detail.

This paper aims to solve a problem that defies solution in this line of research. We consider the class of linear systems with saturated controls and *stochastic disturbances* and address the stabilization question to bound the variance of the state, while simultaneously achieving global asymptotic stability in the absence of disturbances. This problem seems a natural extension of the results in [6], [10]. Specifically, we consider a linear time invariant system subject to input saturation, stochastic external disturbances and random Gaussian distributed initial conditions, independent of the external disturbances. The aim will be to control this system by a possibly nonlinear static state feedback law that achieves global asymptotic stability in the absence of disturbances while guaranteeing a bounded variance of the state vector for all time. We consider this problem both in discrete as well as in continuous time under the mild assumption that the unconstrained system is stabilizable.

It is important to emphasize that in either case it is not clear that such feedbacks will exist. The main results of this paper provide a rather complete solution to this problem. Existence conditions are derived in terms of the plant data and we synthesize an explicit feedback control law for achieving bounded variance in the controlled system. This control law is non-linear and based on a scheduling parameter that can be computed explicitly. We develop the theory and results for both the discrete time as well as the continuous time case separately. Although the main ideas for these two cases are similar, either case has its own technical merit.

The paper is organized as follows. For both discrete time as well as continuous time systems a formal problem formulation and the main results are stated in Section II.

In this conference version only a proof of the discrete-time version will be presented in Section III. In Section IV, the main result is illustrated by the global stabilization of a noise corrupted double integrator by means of a nonlinear feedback controller. A discussion on the implications of the main results, some extensions and conclusions are collected in Section V.

II. PROBLEM FORMULATION AND MAIN RESULTS

A. The discrete time case

In discrete time we consider systems of the form

$$x(t+1) = Ax(t) + Bu(t) + Ew(t) \tag{1}$$

where the state x, the control u and the disturbance w are vector valued signals of dimension n, m and ℓ, respectively. Here, $t \in \mathbb{Z}_+$, w is a white noise stochastic process of unit variance, the initial condition x_0 of (1) is a Gaussian random vector independent of $w(t)$ for all $t \geq 0$. The control input u is constrained in that

$$u(t) \in \mathbb{U}, \qquad t \in \mathbb{Z}_+ \tag{2}$$

where $\mathbb{U} = [-1,1]^m$ is the unit hypercube in $\mathbb{R}^m$. An *admissible feedback* is an expression of the form

$$u(t) = f(x(t)) \tag{3}$$

where $f : \mathbb{R}^n \to \mathbb{U}$ is a continuous map with $f(0) = 0$. We therefore consider nonlinear static state feedbacks. We will be interested in the following problem.

Problem II.1 Given the system (1), the *simultaneous global asymptotic stabilization and bounded variance problem* is to find an admissible feedback (3) such that the following properties hold:

1) in the absence of the external input w, the equilibrium point $x = 0$ of the controlled system (1)-(3) is globally asymptotically stable.
2) for any white noise input w and any Gaussian random initial condition x_0 of (1) that is independent of $w(t)$ for all $t \geq 0$, the variance $\mathrm{Var}(x(t))$ of the controlled system (1)-(3) is bounded for all $t \geq 0$.

The following is the main result of this paper for discrete time systems.

Theorem II.2 *Consider the system* (1) *and suppose that* (A, B) *is stabilizable. Then there exists a feedback* (3) *such that, with* $w = 0$*, the equilibrium point* $x = 0$ *is globally asymptotically stable if and only if all eigenvalues of* A *are inside or on the unit circle. In that case, the simultaneous global asymptotic stabilization and bounded variance problem will be solvable and one admissible feedback that solves problem II.1 is given as follows. There exists* $\gamma > 0$ *such that for all* $\varepsilon \in (0,1]$ *the Riccati equation*

$$\begin{aligned} P = A^\top PA + \varepsilon I - A^\top P \begin{pmatrix} B & E \end{pmatrix} \\ \times \underbrace{\begin{pmatrix} B^\top PB + I & B^\top PE \\ E^\top PB & E^\top PE - \gamma^2 I \end{pmatrix}}_{W}^{-1} \begin{pmatrix} B^\top \\ E^\top \end{pmatrix} PA \end{aligned} \tag{4}$$

admits a unique solution $P = P^\top \geq 0$ *such that*

$$W_{11} := B^\top PB + I > 0 \tag{5a}$$

$$W_{22} := E^\top PE - \gamma^2 I < 0 \tag{5b}$$

$$A_{cl} := A - \begin{pmatrix} B & E \end{pmatrix} W^{-1} \begin{pmatrix} B^\top \\ E^\top \end{pmatrix} PA \text{ is Schur.} \tag{5c}$$

Let this solution be denoted by $P(\varepsilon)$ *and define the scheduling function* $\varepsilon : \mathbb{R}^n \to (0,1]$ *by*

$$\varepsilon(x) := \max\{r \in (0,1] \mid x^\top P(r) x \operatorname{trace} P(r) \leq c\}. \tag{6}$$

Then the composite function $P(\varepsilon(x))$ *with* $x \in \mathbb{R}^n$*, denoting the solution* P *of* (4)-(5) *with* ε *replaced by* $\varepsilon(x)$*, is well defined. Define* $f : \mathbb{R}^n \to \mathbb{R}^m$ *by*

$$f(x) := -[D_{11}(\varepsilon(x))]^{-1} N_{11}(\varepsilon(x)) P(\varepsilon(x)) Ax \tag{7}$$

where

$$\begin{aligned} D_{11}(\varepsilon) := B^\top P(\varepsilon) B + I - B^\top P(\varepsilon) E \\ \times (E^\top P(\varepsilon) E - \gamma^2 I)^{-1} E^\top P(\varepsilon) B \end{aligned}$$

and

$$N_{11}(\varepsilon) := B^\top - B^\top P(\varepsilon) E \left(E^\top P(\varepsilon) E - \gamma^2 I\right)^{-1} E^\top.$$

Then f *is a well-defined admissible feedback* (3) *that solves Problem II.1 for* c *small enough.*

B. The continuous time case

In continuous time we consider the differential equation

$$\mathrm{d}x(t) = Ax(t)\mathrm{d}t + Bu(t)\mathrm{d}t + E\mathrm{d}w(t) \tag{8}$$

where w is a Wiener process (a process of ℓ independent Brownian motions), and the initial condition x_0 of (8) is a Gaussian random vector, independent of $w(\cdot)$. Its solution x is rigorously defined through Wiener integrals and is a Gauss-Markov process whose expected value $\mu(t) = \mathbb{E}(x(t))$ and covariance $\Pi(t) = \mathrm{Cov}(x(t))$ satisfy

$$\begin{aligned} \dot{\mu} &= A\mu + B\sigma(u), & \mu(0) &= \mathbb{E}(x_0) \\ \dot{\Pi}(t) &= A\Pi + \Pi A^\top + B^\top B, & \Pi_0 &= \mathrm{Cov}(x_0). \end{aligned}$$

Similar to the discrete time case, the control input u of (8) is constrained in that

$$u(t) \in \mathbb{U}, \qquad t \in \mathbb{R}_+ \tag{9}$$

where $\mathbb{U} = [-1,1]^m$ is the unit hypercube in $\mathbb{R}^m$. Like in the discrete time case, admissible feedbacks are possibly nonlinear static state feedbacks of the form (3) where $f : \mathbb{R}^m \to \mathbb{U}$ is a Lipschitz-continuous mapping with $f(0) = 0$.

Problem II.3 Given the system (8), the *simultaneous global asymptotic stabilization and bounded variance problem* is to find an admissible feedback (3) such that the following properties hold:

1) in the absence of the external input w, the equilibrium point $x = 0$ of the controlled system (8)-(3) is globally asymptotically stable.
2) for any Wiener process w and any Gaussian random initial condition x_0 of (8) that is independent of $w(t)$ for all $t \geq 0$, the variance $\mathrm{Var}(x(t))$ of the controlled system (8)-(3) is bounded for all $t \geq 0$.

The following is the main result of this paper for continuous time systems.

Theorem II.4 *Consider the system* (8) *and suppose that* (A, B) *is stabilizable. Then there exists a feedback* (3) *such that with* $w = 0$*, the equilibrium point* $x = 0$ *is globally asymptotically stable if and only if all eigenvalues of* A *are in the closed left-half complex plane. In that case, the simultaneous global asymptotic stabilization and bounded variance problem will be solvable and one admissible feedback that solves problem II.3 is given as follows. There exists* $\gamma > 0$ *such that the Riccati equation*

$$PA + A^\top P - PBB^\top P + \gamma^{-2} PEE^\top P + \varepsilon I = 0$$

has a unique positive definite solution $P(\varepsilon)$ *for any* $\varepsilon \in (0, 1]$*. Define the scheduling function* $\varepsilon : \mathbb{R}^n \to (0, 1]$ *by*

$$\varepsilon(x) := \max\{r \in (0,1] \mid (x^\top P(r) x)\,\mathrm{trace}(B^\top P(r) B) \leq 1\}$$

and let $P(\varepsilon(x))$ *denote the solution of the Riccati equation where* ε *is replaced by* $\varepsilon(x)$*. Set*

$$f(x) := -B^\top P(\varepsilon(x)) x.$$

Then f *defines an admissible feedback* (3) *that solves Problem II.3.*

III. PROOFS FOR THE DISCRETE TIME CASE

For $\varepsilon \in (0, 1]$, consider the unconstrained system

$$x(t+1) = Ax(t) + Bu(t) + Ew(t); \quad x(0) = x_0$$
$$z_\varepsilon(t) = \sqrt{\varepsilon} \begin{pmatrix} I \\ 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ I \end{pmatrix} u(t).$$

and let $\mathcal{U}$ be the class of linear feedbacks $u = Fx$ for which $A + BF$ is Schur. Define

$$\gamma^*(\varepsilon) := \inf_{u \in \mathcal{U}} \sup_{0 \neq \|w\| < \infty} \frac{\|z_\varepsilon\|}{\|w\|}$$
$$J_\varepsilon := J_\varepsilon(x_0, u, w) := \|z_\varepsilon\|^2 - \gamma^2 \|w\|^2.$$

where $\|\cdot\|$ denotes the usual 2-norm and where $\gamma^*(\varepsilon)$ is defined only if $x_0 = 0$. Then γ^*_ε is the optimal achievable H_∞ norm of the closed loop system in the class of stabilizing state feedbacks F. The cost $J_\varepsilon(x_0, u, w)$ denotes the value function of a zero sum game with initial condition $x(0) = x_0$ and strategies u and w. Since $\|z_\varepsilon\|$, viewed as function of $\varepsilon \in (0, 1]$, is non-decreasing, it is immediate that $\gamma^*(\varepsilon)$ will be non-decreasing. Hence, $\gamma^*(\varepsilon) \leq \gamma^*(1)$ for all $\varepsilon \in (0, 1]$. Now fix $\gamma > \gamma^*(1)$. Then for all $\varepsilon \in (0, 1]$ there exists a control $u = F(\varepsilon)x$, such that $A + BF(\varepsilon)$ is Schur stable and, in closed loop,

$$\gamma^*(\varepsilon) \leq \sup_{0 \neq \|w\| < \infty} \frac{\|z_\varepsilon\|}{\|w\|} \leq \gamma. \tag{10}$$

Using the state space characterization of achievable H_∞ performance, (see for example [1]), this implies that for all $\varepsilon \in (0, 1]$ there exists a symmetric matrix $P(\varepsilon) \geq 0$ that solves the discrete time Riccati equation (4) under the conditions (5). Simplifying the notation by ignoring all arguments ε, and recalling that

$$W = \begin{pmatrix} B^\top PB + I & B^\top PE \\ E^\top PB & E^\top PE - \gamma^2 I \end{pmatrix} = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix},$$

one such control will be $u^* = Fx$ where

$$F := -\left(W_{11} - W_{12} W_{22}^{-1} W_{21}\right)^{-1} \times \left(B^\top - W_{12} W_{22}^{-1} E^\top\right) PA.$$

This expression is inferred by applying a completion of the squares argument for J_ε, followed by a Schur complement of W and the observation that for all x_0, u and w, the cost $J_\varepsilon = J_\varepsilon(x_0, u, w)$ satisfies

$$\begin{aligned} J_\varepsilon &= x_0^\top P x_0 + \| \begin{pmatrix} u \\ w \end{pmatrix} + W^{-1} \begin{pmatrix} B^\top \\ E^\top \end{pmatrix} PAx \|_W^2 \\ &= x_0^\top P x_0 + \| \begin{pmatrix} u \\ v \end{pmatrix} + D^{-1} N \begin{pmatrix} B^\top \\ E^\top \end{pmatrix} PAx \|_D^2 \\ &= x_0^\top P x_0 + \| \begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} F \\ G \end{pmatrix} x \|_D^2 \end{aligned}$$

where $\|\xi\|_D^2 = \langle \xi, D\xi \rangle$, $v = w + W_{22}^{-1} W_{21} u$, $D = NWN^\top$ and the matrices D, N and G are defined as

$$\begin{aligned} D &:= \begin{pmatrix} W_{11} - W_{12} W_{22}^{-1} W_{21} & 0 \\ 0 & W_{22} \end{pmatrix} \\ N &:= \begin{pmatrix} I & -W_{12} W_{22}^{-1} \\ 0 & I \end{pmatrix} \\ G &:= -W_{22}^{-1} E^\top PA. \end{aligned}$$

Indeed, one can infer from $P > 0$ that $u^* := Fx$ is a stabilizing feedback (i.e., it belongs to $\mathcal{U}$) while $D_{22} = W_{22} < 0$ together with the above expression for J_ε yields that $J_\varepsilon(0, u^*, w) \leq 0$ for all w. This, in turn, implies (10). Now set

$$\begin{aligned} D_{11} &= W_{11} - W_{12} W_{22}^{-1} W_{21} \\ N_{11} &= \begin{pmatrix} I & -W_{12} W_{22}^{-1} \end{pmatrix} \begin{pmatrix} B^\top \\ E^\top \end{pmatrix} \end{aligned}$$

and recall that the latter are functions of ε. Also note that $F = -D_{11}^{-1} N_{11} PA$. We infer some properties of $P(\varepsilon)$, $D_{11}(\varepsilon)$ and $N_{11}(\varepsilon)$ in the following lemma.

Lemma III.1 *There exists $\gamma > 0$ such that the matrices $P(\varepsilon)$, $D_{11}(\varepsilon)$ and $N_{11}(\varepsilon)$ have the following properties:*

1) *$P(\varepsilon)$ is increasing in ε and $\lim_{\varepsilon\downarrow 0} P(\varepsilon) = 0$.*
2) *$P(\varepsilon)$ is continuously differentiable for $\varepsilon \in (0, 1]$.*
3) *there exists a constant $q > 0$ such that*
$$\|P^{1/2}(\varepsilon)AP^{-1/2}(\varepsilon)\| \leq q$$
for any $\varepsilon \in (0, 1]$.
4) *$D_{11}(\varepsilon) := W_{11}(\varepsilon) - W_{12}(\varepsilon)W_{22}^{-1}(\varepsilon)W_{21}(\varepsilon)$ is non-singular and satisfies $D_{11}^{-1}(\varepsilon) \leq I$ uniformly over $\varepsilon \in (0, 1]$.*
5) *$\lambda_{\max}(N_{11}^\top(\varepsilon)N_{11}(\varepsilon)) \leq 4\lambda_{\max}(BB^\top)$ uniformly over $\varepsilon \in (0, 1]$.*

Proof: We already proved that there exists $\gamma > 0$ such that $P(\varepsilon)$, $D(\varepsilon)$ and $N(\varepsilon)$ are well defined for all $\varepsilon \in (0, 1]$.

1) Note that J_ε is increasing in ε for fixed u and w. Therefore, since

$$J_\varepsilon^*(x) = \inf_u \sup_w J_\varepsilon(x, u, w) = x^\top P(\varepsilon)x$$

implies that P_ε is increasing in ε. If $\lim_{\varepsilon\downarrow 0} P(\varepsilon) \neq 0$ there will exist a non-zero x such that $\lim_{\varepsilon\downarrow 0} x^\top P(\varepsilon)x = 1$ (note that $P(\varepsilon)$ is increasing and hence the limit always exists). Note that $J_\varepsilon^*(x)$ introduced above is the Nash equilibrium value of the game with initial condition $x(0) = x$ under state feedback strategies (u, w). This immediately implies that $\lim_{\varepsilon\downarrow 0} J_\varepsilon^*(x) = 1$. If equilibrium feedback strategies $(u_\varepsilon^*, w_\varepsilon^*)$ exist, then $J_\varepsilon^*(x) \leq J_\varepsilon(x, u, w_\varepsilon^*)$ for all u. In particular, with $u = 0$ it follows that

$$J_\varepsilon^*(x) \leq J_\varepsilon(x, 0, w_\varepsilon^*) = \varepsilon\|x\|^2 - \|w_\varepsilon^*\|^2 \leq \varepsilon\|x\|^2.$$

Taking limits $\varepsilon \to 0$ this yields

$$1 = \lim_{\varepsilon\to 0} J_\varepsilon^*(x) \leq \lim_{\varepsilon\to 0} J_\varepsilon(x, 0, w_\varepsilon^*) \leq 0$$

which yields a contradiction. If equilibrium strategies do not exist, a similar reasoning on 'almost equilibria' will also result in a contradiction. Conclude that $\lim_{\varepsilon\to 0} P(\varepsilon) = 0$.

2) Follows from the standard continuity arguments for solutions of Riccati equations.

3) Pre- and post- multiplying the Riccati equation (4) by $P^{-1/2}$ yields that

$$P^{-1/2}A^\top P^{1/2}\left[I - P^{1/2}\begin{pmatrix}B & E\end{pmatrix}W^{-1}\begin{pmatrix}B^\top \\ E^\top\end{pmatrix}P^{1/2}\right] \times P^{1/2}AP^{-1/2} = I - \varepsilon P^{-1} \leq I$$

By item 1, $\lim_{\varepsilon\downarrow 0} P(\varepsilon) = 0$, and hence the expression between square brackets converges to I as $\varepsilon \to 0$. Hence, there exists ε_1 such that for all $\varepsilon \in (0, \varepsilon_1]$ we have

$$P^{-1/2}(\varepsilon)A^\top P(\varepsilon)AP^{-1/2}(\varepsilon) \leq 2I - 2\varepsilon P^{-1} \leq 2I$$

which shows that $\|P^{-1/2}(\varepsilon)AP^{1/2}(\varepsilon)\| \leq \sqrt{2}$ uniformly on $(0, \varepsilon_1]$. Moreover, $P(\varepsilon)$ is bounded from above and bounded from below away from 0 on the interval $[\varepsilon_1, 1]$ and hence there exists q such that item 3 is satisfied.

4) Since $W_{22}(\varepsilon) < 0$ and $P(\varepsilon) > 0$ implies $W_{11}(\varepsilon) = B^\top P(\varepsilon)B + I \geq I$ we have that

$$D_{11} = W_{11} - W_{12}W_{22}^{-1}W_{21} \geq W_{11} \geq I.$$

Consequently, $D_{11}^{-1}(\varepsilon) \leq I$ uniformly on $(0, 1]$.

5) For $\gamma > \gamma^*(1)$, we already showed that (4) admits a solution $P = P(\varepsilon, \gamma) > 0$ satisfying (5) for all $\varepsilon \in (0, 1]$. In particular, multiplying the inequality (5b) by 2 yields that

$$\left(\sqrt{2}\gamma\right)^2 - 2E^\top P(\varepsilon, \gamma)E > 0$$

With $\gamma' = \sqrt{2}\gamma$ we then have

$$(\gamma')^2 > 2E^\top P(\varepsilon, \frac{1}{2}\sqrt{2}\gamma')E > 2E^\top P(\varepsilon, \gamma')E$$

where the last inequality follows from the monotonicity of $P(\varepsilon, \cdot)$ in that $\gamma < \gamma'$ implies $P(\varepsilon, \gamma) > P(\varepsilon, \gamma')$. We thus conclude that we can find γ^* such that for $\gamma > \gamma^*$ we have that $\gamma^2 - 2E^\top P(\varepsilon, \gamma)E > 0$ for all $\varepsilon \in (0, 1]$. We infer that for all $\varepsilon \in (0, 1]$:

$$\begin{aligned}
& 2E^\top PE < \gamma^2 I \\
\Rightarrow\ & E^\top PE < \gamma^2 I - E^\top PE \\
\Rightarrow\ & E^\top PE < (\gamma^2 I - E^\top PE)^{1/2}(\gamma^2 I - E^\top PE)^{1/2} \\
\Rightarrow\ & (\gamma^2 I - E^\top PE)^{-1/2}E^\top PE(\gamma^2 I - E^\top PE)^{-1/2} < I \\
\Rightarrow\ & P^{1/2}E(\gamma^2 I - E^\top PE)^{-1}E^\top P^{1/2} < I \\
\Rightarrow\ & \|PE(\gamma^2 I - E^\top PE)^{-1}E^\top\| < 1 \\
\Rightarrow\ & \|I - PE(\gamma^2 I - E^\top PE)^{-1}E^\top\|^2 < 4 \\
\Rightarrow\ & \|B^\top(I - PE(\gamma^2 I - E^\top PE)^{-1}E^\top)\|^2 \leq 4\|B^\top\|^2 \\
\Rightarrow\ & \|N_{11}\|^2 \leq 4\|B^\top\|^2 \\
\Rightarrow\ & \lambda_{\max}(N_{11}^\top N_{11}) \leq 4\lambda_{\max}(B^\top B)
\end{aligned}$$

Hence, $\lambda_{\max}(N_{11}(\varepsilon)^\top N_{11}(\varepsilon)) \leq 4\lambda_{\max}(BB^\top)$ uniformly for $\varepsilon \in (0, 1]$. This yields the claim. ∎

Choose γ according to the hypothesis of Lemma III.1 and note that $f(x)$ defined in (7) is equal to Fx. Consider the scheduling function (6) with

$$c := \frac{1}{4q^2\lambda_{\max}(BB^\top)}. \tag{11}$$

By item 4 of Lemma III.1, $D_{11}(\varepsilon)$ is non-singular for all $\varepsilon \in (0, 1]$. Consequently, also $D_{11}(\varepsilon(x))$ will be non-singular for all $x \in \mathbb{R}^n$. Hence, f defined in (7) is a well defined function on $\mathbb{R}^n$. Moreover, using the properties of Lemma III.1 we infer that for any $x \in \mathbb{R}^n$:

$$\begin{aligned}
\|f(x)\|^2 &= x^\top A^\top PN_{11}^\top D_{11}^{-2}N_{11}PAx \\
&\leq \|P^{1/2}AP^{-1/2}\|^2\|N_{11}^\top D_{11}^{-2}N_{11}\|x^\top Px \operatorname{trace} P \\
&\leq q^2\lambda_{\max}(N_{11}^\top N_{11}\lambda_{\max}(D_{11}^{-2})x^\top Px \operatorname{trace} P \\
&\leq 4q^2\lambda_{\max}(BB^\top)c \\
&\leq 1.
\end{aligned}$$

Hence, $f : \mathbb{R}^n \to \mathbb{U}$ is an admissible feedback and it follows that the control (3) achieves that $u(t) \in \mathbb{U}$ for all $t \geq 0$.

Next, we show that the closed loop system is globally asymptotically stable when $w = 0$. For this, consider the function

$$V(x) = x^\top P(\varepsilon(x))x$$

and, simplifying and abusing notation, let us define $V(t) := V(x(t))$, $P(t) := P(\varepsilon(x(t)))$ and $\varepsilon(t) := \varepsilon(x(t))$. A number of observations need to be made. Firstly, observe that $P(t) > 0$ implies $V(t) > 0$ for any $t \geq 0$. Secondly, infer from the definition of the scheduling function that for all t we have $V(t) \operatorname{trace} P(t) \leq c$. Thirdly, item 1 of Lemma III.1 implies that $V(t) \operatorname{trace} P(t) = c$ whenever $\varepsilon(t) < 1$. Fourthly, using the (pointwise in time) expressions for J_ε, the variations of V along state trajectories of the controlled system (1)-(3) satisfy

$$\begin{aligned} V(t+1) - V(t) = x^\top(t)[P(t+1) - P(t)]x(t) \\ - \varepsilon(t)\|x(t)\|^2 - \|u(t)\|^2 + \gamma^2\|w(t)\|^2 \\ + \|v(t) - Gx(t)\|^2_{W_{22}} \end{aligned} \tag{12}$$

where, as stated before, $v = w + W_{22}^{-1}W_{21}u$. Setting $w = 0$ and noting that $W_{22} = E^\top PE - \gamma^2 I < 0$, this yields that

$$\begin{aligned} V(t+1) - V(t) \leq x(t)[P(t+1) - P(t)]x(t) \\ - \varepsilon(t)\|x(t)\|^2. \end{aligned} \tag{13}$$

Now, two situations may occur: If $\varepsilon(x(t+1)) \leq \varepsilon(x(t))$ then $P(t+1) \leq P(t)$ and it follows from (13) that $V(t+1) - V(t) < 0$ provided that $x(t) \neq 0$. If, on the other hand, $1 \geq \varepsilon(x(t+1)) > \varepsilon(x(t))$ then we must have that $P(t+1) > P(t)$ and

$$\begin{aligned} V(t) \operatorname{trace} P(t) = x^\top(t)P(t)x(t) \operatorname{trace} P(t) = c \\ \geq V(t+1) \operatorname{trace} P(t+1) \end{aligned}$$

This implies that $V(t+1) - V(t) < 0$ provided that $x(t) \neq 0$. In either case $V(t)$ is strictly decreasing along (non-zero) state trajectories of the controlled system with $w = 0$. Conclude that V is a Lyapunov function and, consequently, with $w = 0$, the controlled system (1)-(7) will be globally asymptotically stable.

The main step that remains is to establish that the variance of the state remains bounded. This can be approached by analyzing the behavior of $\mathbb{E}\, V(t)$. It can be established through tedious and lengthy analysis that $\mathbb{E}\, V(t)$ becomes decreasing for $V(t)$ large enough due to the fact that the size of the disturbances needed to make V increasing grow with the size of V itself. The Gaussian nature of the noise make the probability of such w exponentially decaying with the size of $V(t)$. Note that the analysis of (12) shows that the nonstandard term due to the fact that the Lyapunov function contains a scheduling parameter is the term $x^\top(t)[P(t+1) - P(t)]x(t)$. The scheduling is such that V is increasing if and only if P is decreasing and hence when this nonstandard term is negative.

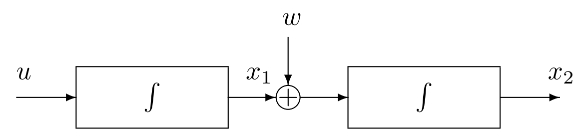

Fig. 1. Double integrator system

IV. EXAMPLE

In this section we will illustrate the use of Theorem II.2 by solving the simultaneous global asymptotic stabilization and bounded variance problem (formulated in Section II) for a double integrator system illustrated in Figure 1. The system is corrupted by a white noise stochastic process w of unit variance. u is the control input that is required to assume values $u(t) \in \mathbb{U} := [-0.5, 0.5]$.

The system has been discretized (zero order hold) with unit sampling rate $T_s = 1$, leading to a non-stable state space representation (1) with

$$A = \begin{pmatrix} 1 & 0 \\ T_s & 1 \end{pmatrix}; \quad B = \begin{pmatrix} T_s \\ 0 \end{pmatrix}; \quad E = \begin{pmatrix} 0 \\ T_s \end{pmatrix}.$$

With $\gamma = 2.24$, the solution $P(\varepsilon)$ of (4) and (5) exists for all $\varepsilon \in (0, 1]$ (as promised by Theorem II.2). To meet the input amplitude constraints, the constant c in the scheduling function (6) is set to $c = 0.5$. (See also (11). The system is controlled by the feedback $u(t) = f(x(t))$ defined in (7). The gain of the nonlinear feedback f on $[-2, 2] \times [-2, 2]$ is illustrated in Figure 2.

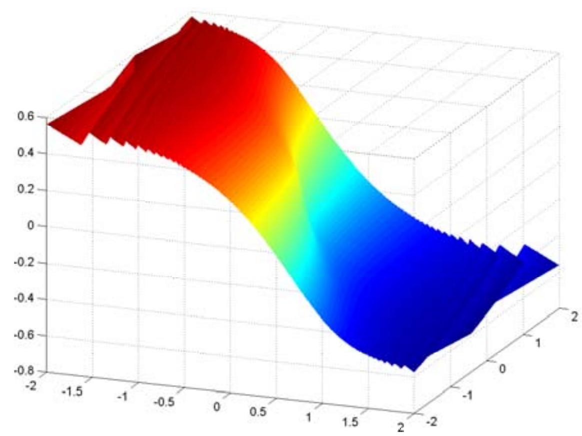

Fig. 2. Nonlinear stabilizing feedback $f : [-2, 2] \times [-2, 2] \to \mathbb{U}$

In the absence of noise, the controlled system is globally asymptotically stable. With w a white noise process of unit variance and x_0 a random initial condition, a time simulation (100 samples) of the scheduling function $\varepsilon(x(t))$, the control input $u(t)$ and the Lyapunov function $V(t) := x(t)^\top P(\varepsilon(x(t)))x(t)$ in the closed loop system are plotted in Figure 3. As proven in Section III, the Lyapunov function is strictly decreasing if $w = 0$. For non-zero noise, $V(t)$ is a stochastic process of bounded expectation. Observe that the scheduling function is actively adapting the control input.

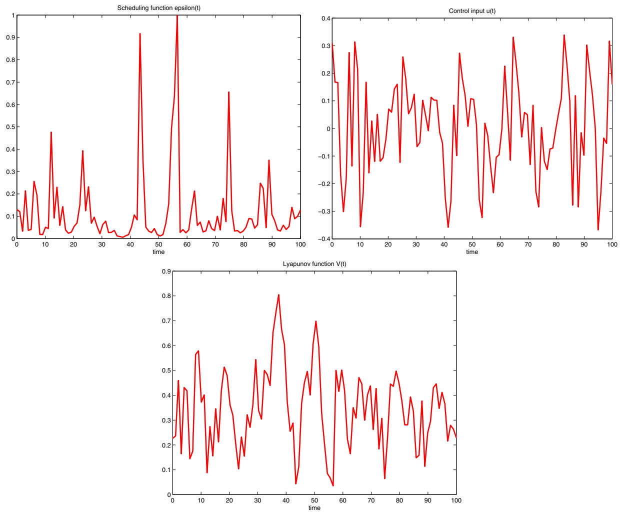

Fig. 3. 100 time samples of the scheduling function (top-left), the control inputs (top-right) and Lyapunov function of the noise corrupted closed loop system.

V. DISCUSSION AND CONCLUSIONS

In this paper we considered the problem of finding a stabilizing controller for a linear time invariant system subject to input constraints and stochastic disturbances. Using a scheduled linear controller, it is shown that in the absence of external disturbances global stabilization of the system can be achieved while simultaneously guaranteeing the state of the controlled system to have bounded variance. The main result is presented for both discrete and continuous time systems and constitutes a natural generalization of concepts such as ISS introduced by Sontag to systems with stochastic disturbances. We established a notion of stability in which the state variance is bounded.

Remark V.1 The constrained set $\mathbb{U}$, defined in this paper as the unit hypercube $[-1, 1]^m$, can be easily generalized to other constrained sets, possibly implying an adaptation of the constant c in Theorem II.2. Note that the constant c in the main theorems has been defined explicitly in (11).

Remark V.2 An interesting question for further research amounts to minimizing the state variance.

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